

DT#9 1

Homomorphisms Casson invariant

X - c.g. 3-fold $M = H_L =$ mod sp. of ^{semi} stable sheaves
w/ fixed rk, det. & Chern #'s.

Assume $\deg$ & rk are coprime.

semistable $\Rightarrow$ stable.

Thomas: M has a perfect obstruction theory

Means: $\exists E_1, E_2 \rightarrow M$, vector bundles.

s.t. for $[\xi_0] \in M$, $\exists \mathcal{O}_S$ -mod on $X \times S$

$f: S \rightarrow M \quad \exists$ s.e.s.

$$0 \rightarrow T'_{\xi_0}(\mathcal{I}) \rightarrow \underline{f^* \xi_0 \otimes \mathcal{I}} \rightarrow f^* \xi_0 \otimes \mathcal{I} \rightarrow T^2_{\xi_0} \otimes \mathcal{I} \rightarrow 0$$

v.b./s, but rk can drop.

Similar to index theorem: stabilize!

Tangent
function

$$T'_{\xi_0}(\mathcal{I})_0 = \mathcal{O}_S - \text{mod.} \quad \text{where global sections are isom. classes}$$

$$= \text{Ext}^1_p(\xi_0, \xi_0 \otimes f^* \mathcal{I})_0$$

$$E \rightarrow X \times (S \times \mathcal{I})$$

$$\text{s.t. } E|_{X \times S} = \xi_0.$$

$X \times S \rightarrow S$, derived of f & Hom

Obsta.
function

$$T^2_{\xi_0}|_0 = \text{Ext}^2_p(\xi_0, \xi_0)_0$$

Point being:

$$\underbrace{S \subset Y \subset Y}_n \underbrace{\quad}_m$$

$$0 \rightarrow \mathcal{O} \xrightarrow{\mathcal{O}} \mathcal{O} \xrightarrow{\mathcal{O}} \mathcal{O} \rightarrow 0$$

Apply $\text{Hom}(\xi_0, -)$

$$\text{Ext}^1(\xi_0, \xi_0 \otimes M) \rightarrow \text{Ext}^1(\xi_0, \xi_0 \otimes N) \xrightarrow{2} \text{Ext}^2(\xi_0, \xi_0 \otimes J).$$

Ob.

E_1 and E_2 defined using quot. construction
 (turn out loc. free if $\text{Ext}^i(\mathcal{E}_0, \mathcal{E}_0)$ has const rank for $i \geq 3$.) $p: X \times M \rightarrow M$

Choose $0 \rightarrow K \rightarrow p^*(p_* \mathcal{E}(u_1))(-u_1) \rightarrow \mathcal{E} \rightarrow 0$
 u_1 s.t. $\mathcal{E}(u_1)$ glob. gen. and $R^i p_* \mathcal{E}(u_1) = 0 \forall i \geq 1$.
 u_2 s.t. $K(u_2) = \dots = \dots$

$$E_2 = \text{Hom}(K, \mathcal{E}(u_2))$$

$$E_1 = (p_* K(u_2))^* \otimes p_* \mathcal{E}(u_2) / \text{end}(p_* \mathcal{E}(u_1))$$

Li and Tian '98: if M has P.O.T. there exists a virtual moduli cycle

$$[\mathbb{Z}] \subseteq A_{\text{vd}}(M)$$

Here:
 $\text{vd} = \sum (-1)^i \text{Ext}^i(\mathcal{E}, \mathcal{E})$
 $= 0$.

which is "functorial" and $\mathbb{Z} = M$ if $\text{ob} = 0$.
 if $\text{ob} \neq 0$: $[\mathbb{Z}] = c_{\text{top}}(\text{ob})|_M$.
 and M gpt

Definition: The int. Casson (DT) inv't

$$\lambda_{\text{Cass}}(X) = \text{length}(\mathbb{Z}_0) \in \mathbb{N}_L$$

(degree?)

Need to show invariant under deformations of complex structure of X . (check Li-Tian setup).

let $X = X_0$
 $i: \{X_t\} \hookrightarrow \mathcal{X}$ smooth family
 $\downarrow$
 C

Consider the "stable sheaf" $i_* E$ on $\mathcal{X}$.

This has a moduli space $\mathcal{M}$ consisting only of sheaves of the form $(i_t)_* E$, so stay this way by stability.

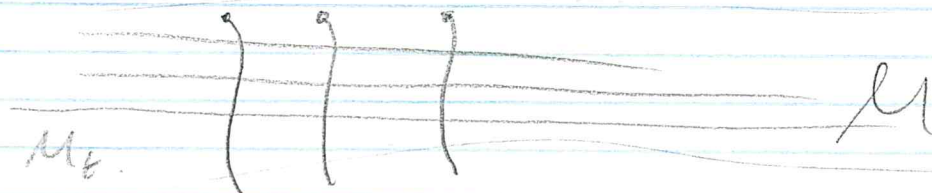
$$\mathcal{M}_t \rightarrow \mathcal{M}$$

$$\downarrow$$

$$C$$

MAIN THM:

Cor. 5.3: $\mathcal{M}$ has a virtual cycle $[Z]$ with
 $\dim Z = \dim \mathcal{M}_t + 1$, and $i_t^! Z = Z_t$



By conservation of intersection # (Fulton),
 $\text{length}(Z_t) = \lambda(X_t)$ is invariant.

Need to check: ① $\mathcal{M}$ has P.O.T., so get Z

② $\mathcal{M}$ and $\mathcal{M}_t$ are "compatible" in sense of L-T,

so $i_t^! Z = Z_t$ as cycles.

Reall Luigi: Lemma 3.42:

$X_9 \in \mathcal{H}$ Center:

- let $E_0 \rightarrow X_0$, $\mathcal{F}_0 = \iota_{X_0}^* E_0 \rightarrow \mathcal{K}$.

Theorem:

$$0 \rightarrow \text{Ext}_{X_0 \times S}^1(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathcal{I}) \rightarrow \text{Ext}_{X \times S}^1(\mathcal{F}_0, \mathcal{F}_0 \otimes \mathcal{I}) \xrightarrow{\text{filter}} \text{Ext}_{X \times S}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes \mathcal{I}) \xrightarrow{\text{trace}} 0$$

filter

trace

$H^0(\mathcal{I})$ if $\mathcal{E}_0$ sample.

$\Rightarrow \mathcal{J}$ is the determinant to deforming $T_0 = i_x \mathcal{E}_0$ in the t direction.

Cor 3.48: Tan-Obs. functors of E_0 and F_0 are abtd by:

$$0 \rightarrow T_{\mathcal{C}_0}'(\mathcal{I}) \hookrightarrow T_{\mathcal{F}_0}'(\mathcal{I}) \rightarrow \mathcal{I} \rightarrow T_{\mathcal{C}_0}' \otimes \mathcal{I} \rightarrow T_{\mathcal{F}_0}' \otimes \mathcal{I} \rightarrow 0$$

have more def. of Γ .

Needed to check: ① F_0 has P.O.T.

① het $m=5$, $\theta = \theta_m$ in Con:

Comb one:

$$0 \rightarrow T'_{\mathcal{F}_0}(I) \rightarrow (E, \oplus \mathcal{O}_N) \otimes I \rightarrow E \otimes I \rightarrow T'_{\mathcal{F}_0} \otimes I \rightarrow 0$$

with less obstructions.

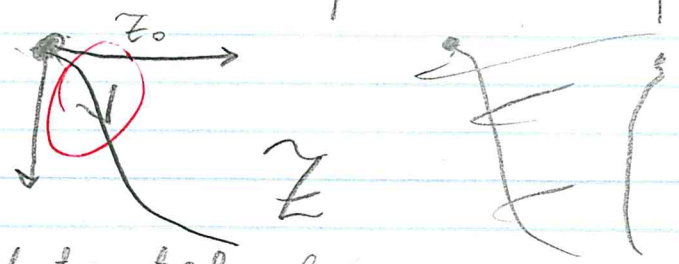
Notice on parties - unilaterals - multilaterals
→ IMPLICIT FCN TERM

By Matthew J. F.

$\frac{1}{\epsilon_0} \rightarrow \frac{1}{\epsilon_0}$

Theorem 3.51

② Also need check a compat. cond. of L-T of the picture worked.



if Z_0 is unobstructed, obvious you can deform it!

The family Z_t is unobstructed w.r.t t (triv.)

$$0 = \text{ob}_{Z_t} = \text{ob}_{Z_0} + \text{ob}_t$$

(So kind as fam. param. by $S \dots$)

$$\Rightarrow \boxed{\text{ob}_{Z_0} = - \text{ob}_t}$$

Necessary for picture to make sense.

(also sufficient for $!Z = Z_t$, Li-Tian)

$$S \subset Y \subset Y,$$

$$\begin{aligned} \mathcal{E} &\rightarrow X_0 \times Y \\ \mathcal{F} &= \tau_* \mathcal{E} \rightarrow X \times Y \end{aligned}$$

$$\text{ob}_{Z_0} \in T_{Z_0}^* \otimes I = \text{Ext}^2(\mathcal{E}_0, \mathcal{E}_0 \otimes I)$$

Also $\text{ob}_t \in \text{Im}(\delta)$

So at least two dir's are on same space.

Claim: if the obstruction $\mathcal{O}(e)$ ^{ob_{Z_0}} vanishes in $\text{Ext}^2(\mathcal{F}_0, \mathcal{F}_0 \otimes I)$

$$\text{then } \mathcal{O}(e) = -\delta(c) = \text{ob}_t$$

Obvious? ^{at least for $n=1$} but need to do at level of functors, see it's always true.

Assume $\partial_0(e) = 0$. $e \in \text{Ext}^1(\xi_0, \xi_{\otimes n}) \xrightarrow{\partial_0} \text{Ext}^1(\xi_0, \xi_{\otimes 0})$

$\Rightarrow$ left f_5 to $\left| \begin{array}{c} l_7 \end{array} \right|$ $\left| \begin{array}{c} l_7 \end{array} \right|$

$$f \in \text{Ext}^1(F_0, F^{\otimes n}) \longrightarrow \text{Ext}^1(F_0, F^{\otimes n}) \longrightarrow \text{Ext}^2(F_0, F^{\otimes n})$$

C

↓ φ

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$$0 \rightarrow \text{Hom}(E_0, E_0 \otimes I) \rightarrow \text{Hom}(E_0, E_0 \otimes W) \rightarrow \text{Hom}(E_0, E_0 \otimes U)$$

A hand-drawn diagram on lined paper. It features a vertical line with a downward-pointing arrow at the bottom. To the right of the line, there is a large, open loop that starts from the top of the line, goes up and around, and then comes back down towards the bottom of the line.

$$E \propto 1 / (\epsilon_0 \epsilon_r I)$$

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Claim: $I_0(c) = -I(c)$

= compat. cond. of L-T.

Horiz. segs: usual Hom₂

Vertical seg's:

$$A_I = \text{opx giving } E_{xt}(\epsilon_0, \epsilon_0 I)$$

$$B_I = \text{rational Ext}^*(J_0, J_0 \otimes I)$$

$$C_T = \text{user! Ext}^{-1}(g, \text{Ext} \circ I)$$

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A_1 & \rightarrow & A_n & \rightarrow & A_n \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

[illegible]

$$\begin{aligned} \gamma(i) &= \gamma(\bar{i}) + \gamma(j|d) = 0 \quad \text{by assumption} \\ &= \gamma(i|\beta) + \gamma(j|d) \Rightarrow \gamma(i|\beta) = -\gamma(j|d) \end{aligned}$$